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NATURAL OSCILLATIONS OF A RECTANGULAR PARALLELEPIPED

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The natural oscillations of a rectangular parallelepiped, which has all its dimensions of similar magnitude, are examined. The numerical results for rectangular bars obtained as a particular case of a general solution show that even the first approximation gives results which are in good agreement with experimental data.

At present, the mathematical theory of elasticity has been used to study only the theory of oscillations of very thin bars and plates and natural oscillations of a sphere. Attempts to formulate a theory of natural oscillations for a right circular cylinder ([1—4], et al.) turned out to be unsuccessful There is no information in the literature on the calculations of the natural oscillations of shapes other than cylinders or spheres with all the dimensions of the same order of magnitude.

1. The reduction of the equations of motion to Helmholtz equations. The equation of motion to be integrated has the form

$$\mu\Delta\mathbf{u} + \rho\omega^2\mathbf{u} + (\lambda + \mu)\operatorname{grad}\operatorname{div}\mathbf{u} = 0 \tag{1.1}$$

where $\mathbf{u}$ is the displacement eigenvector, λ, μ are Lamé elastic constants, ρ is the density of the material, ω is the natural frequency of oscillation.

The boundary conditions corresponding to zero surface loads which act on the solid have the form

$$\sigma\cdot\mathbf{n} = 0 \tag{1.2}$$

where σ is the stress tensor, $\mathbf{n}$ is the normal vector drawn outward from the solid surface.

Equation (1.1) is equivalent to three scalar equations which represent the projections onto the axes of a Cartesian system of coordinates xyz. After differentiating the first of these equations with respect to x, the second with respect to y, and the third with respect to z, and adding the resulting equations, we obtain an equation for the divergence of the displacement eigenvector

$$\Delta\delta + k^2\delta = 0, \quad k^2 = \rho\omega^2 / \lambda + 2\mu \quad (\delta = \operatorname{div}\mathbf{u}) \tag{1.3}$$

On finding the general solution of Eq. (1.3) and substituting it into Eq. (1.1), we obtain a system of nonhomogeneous equations. The general solution of this system can be written in the form

$$\mathbf{u} = \mathbf{u}' - \frac{1}{k^2}\operatorname{grad}\delta \tag{1.4}$$

where the first term is the solution of the homogeneous equation

$$\Delta\mathbf{u}' + k'^2\mathbf{u}' = 0, \quad k'^2 = \frac{\rho\omega^2}{\mu} \tag{1.5}$$

The second term, however, represents the particular integral of the nonhomogeneous equation (1.1) which can be verified from the substitution.

Let us introduce the auxiliary functions f_i $(i = 1, 2, 3)$ which like $\mathbf{u}'$ satisfy the Helmholtz equation (1.5) and, are connected with the components of function $\mathbf{u}'$ by the relations $u_i' = \partial f_i / \partial x_i$. The functions f_i determined in this way are connected by the following relation

$$\sum_{i=1}^{3} \frac{\partial^2 f_i}{\partial x_i^2} = 0 \tag{1.6}$$

Let the parallelepiped be bounded by the planes $x = \pm x_0$, $y = \pm y_0$ and $z = \pm z_0$. In this case, the boundary conditions (1.2) reduce to vanishing of the corresponding components of the tensor σ at the faces of the parallelepiped.

Thus, the problem of natural oscillations in a parallelepiped is reduced to the solution of four Helmholtz equations with respect to three functions f_i and a function δ with boundary conditions (1.2), where the components of the stress tensor must be expressed in terms of the functions f_i and δ.

2. Integration of Helmholtz equations. Let us integrate Eqs. (1.3) and (1.5) using the method of overdetermined series [5]. The main point of the method is that a fairly large class of solutions in the form of sums of series of complete systems of eigenfunctions exist for the elliptic equation which can be solved by the method of separation of variables. The coefficients of these series are the coefficients of the expansion of the unknown solution on the corresponding surfaces. The solutions of ordinary differential equations into which the initial equation breaks down when it is solved by separation of variables are used as eigenfunctions. Thus, for a rectangular parallelepiped, the solution consists of three double series

$$\begin{aligned} \delta(x, y, z) = {} & \sum_{n, m} (A_{nm} \cos \tau_{nm} z + B_{nm} \sin \tau_{nm} z) \sin \frac{n\pi (x + x_0)}{2x_0} \times \\ & \cos \frac{m\pi (y + y_0)}{2y_0} + \sum_{m, l} (C_{ml} \cos \varkappa_{ml} x + D_{ml} \sin \varkappa_{ml} x) \cos \frac{m\pi (y + y_0)}{2y_0} \times \\ & \cos \frac{l\pi (z + z_0)}{2z_0} + \sum_{n, l} (E_{nl} \cos \sigma_{nl} y + F_{nl} \sin \sigma_{nl} y) \times \\ & \sin \frac{n\pi (x + x_0)}{2x_0} \cos \frac{l\pi (z + z_0)}{2z_0} \end{aligned} \tag{2.1}$$

$$\begin{aligned} \varkappa_{ml}^2 &= k^2 - \left(\frac{m\pi}{2y_0}\right)^2 - \left(\frac{l\pi}{2z_0}\right)^2 \\ \sigma_{nl}^2 &= k^2 - \left(\frac{n\pi}{2x_0}\right)^2 - \left(\frac{l\pi}{2z_0}\right)^2; \quad n = 1, 2, 3, \ldots \\ \tau_{nm}^2 &= k^2 - \left(\frac{n\pi}{2x_0}\right)^2 - \left(\frac{m\pi}{2y_0}\right)^2; \quad l, m = 0, 1, 2, \ldots \end{aligned} \tag{2.2}$$

If we use an analogous representation for the functions f_i with the conditions (1.6) and substituting them into (1.4), we obtain a solution of Eq. (1.1) for certain restrictions on the edges of the parallelepiped. These restrictions can be removed by adding the Helmholtz equation to the investigated solution: (1) the solution in the finite form which contains at least 8 arbitrary constants necessary to remove the restrictions at the vertices, and (2) the solution in the form of a set of 12 single series necessary to remove the restrictions at the internal points of the ribs of the parallelepiped. In this case, all the natural oscillations of the free parallelepiped will be found. Otherwise, only a number of all the studied oscillations will be obtained.

In this paper, the results obtained, assuming that δ is represented by (2.1), are described. In addition, the symmetry elements of the examined parallelepiped are made use of. The three symmetry planes $x = 0$, $y = 0$ and $z = 0$ are just such elements. Further we shall examine only the solutions symmetrical with respect to these planes. For this, let us assume that $B_{nm} = D_{ml} = F_{nl} = 0$, $n = 1, 3, 5, \ldots$; $l, m = 0, 2, 4$. For the functions f_i, we use an expression which differs from (2.1) in that the expansion will have superscripts 1, 2, 3, and $\varkappa_{ml}$, τ_{nm}, σ_{nl} will be dashed.

3. Equations for determining the coefficients. The solutions obtained of type (2.1) will have for δ and f_i 12 infinite sequences of unknown coefficients which should be determined from the boundary conditions (1.2) and the relation (1.6). It is obvious that condition (1.6) will be satisfied if the coefficients are connected by the relations

$$\left(\frac{n\pi}{2x_0}\right)^2 A_{nm}^{(1)} + \left(\frac{m\pi}{2y_0}\right)^2 A_{nm}^{(2)} + \tau_{nm}^{\prime 2} A_{nm}^{(3)} = 0 \tag{3.1}$$

$$\varkappa_{ml}^{\prime 2} C_{ml}^{(1)} + \left(\frac{m\pi}{2y_0}\right)^2 C_{ml}^{(2)} + \left(\frac{l\pi}{2z_0}\right)^2 C_{ml}^{(3)} = 0$$

$$\left(\frac{n\pi}{2x_0}\right)^2 E_{ml}^{(1)} + \sigma_{nl}^{\prime 2} + E_{nl}^{(2)} + \left(\frac{l\pi}{2z_0}\right)^2 E_{nl}^{(3)} = 0$$

The remaining nine relations are obtained from the boundary conditions in the following way. We write the expression for the component of the stress tensor in terms of the solutions constructed for δ and f_i, and represent each of these expressions by a unique Fourier series. Then, on satisfying the homogeneous boundary conditions, we equate the coefficients in the expansion of the Fourier series to zero. Thus, requiring that the components of the stress tensor τ_{xy} on the face $x = x_0$ be zero

$$\tau_{xy} = \mu \frac{\partial^2}{\partial x\, \partial y}\left(f_1 + f_2 - \frac{2}{k^2}\delta\right)$$

expanding the expression obtained into a Fourier series in terms of the functions

$$\cos\frac{m\pi(y + y_0)}{2y_0} \cos\frac{l\pi(z + z_0)}{2z_0}$$

and equating all expansion coefficients to zero, we obtain

$$\begin{aligned}
&\varkappa_{ml}^{\prime} \operatorname{tg} \varkappa_{ml}^{\prime} x_0 \left(C_{ml}^{(1)} + C_{ml}^{(2)}\right) - \frac{2}{k^2} \varkappa_{ml} \operatorname{tg} \varkappa_{ml} x_0 \times \\
&\left(C_{ml} + \frac{2}{z_0} \sum_n^{l} \left(\frac{n\pi}{2x_0}\right) \left[\tau_{nm}^{\prime} \left[\tau_{nm}^{\prime 2} - \left(\frac{l\pi}{2z_0}\right)^2\right]^{-1} \left(A_{nm}^{(1)} + A_{nm}^{(2)}\right) - \right.\right. \\
&\left.\frac{2}{k^2} \tau_{nm} \left[\tau_{nm}^2 - \left(\frac{l\pi}{2z_0}\right)^2\right]^{-1} A_{nm}\right] + \\
&\frac{2}{x_0} \sum_n \left(\frac{n\pi}{2x_0}\right) \left[\sigma_{nl}^{\prime} \left[\sigma_{nl}^{\prime 2} \left(\frac{m\pi}{2y_0}\right)^2\right]^{-1} \left(E_{nl}^{(1)} + E_{nl}^{(2)}\right) - \right. \\
&\left.\left.\frac{2}{k^2} \sigma_{nl} \left[\sigma_{nl}^{\prime 2} - \left(\frac{m\pi}{2y_0}\right)^2\right]^{-1} E_{nl}\right] = 0, \quad l, m = 0, 2, 4, \ldots
\end{aligned} \tag{3.2}$$

In the same way, by equating the remaining eight components of the stress tensor to zero, we obtain eight further relations between the unknown sequences of the coefficients.

It should be noted that from the 12 relations between the unknowns, eight are simple, i.e. of the type (3.1), and the remaining four — of the type (3.2). This makes it fairly easy to reduce the problem to two infinite systems with respect to two infinite sequences of unknowns A_{nm} and E_{nl}

$$\sum_{i,l}^{m} P^{\alpha}_{nilm} E_{il} + \sum_{i} T^{\alpha}_{nim} A_{im} + \sum_{l} K^{\alpha}_{nml} E_{nl} + L^{\alpha}_{nm} A_{nm} = 0 \qquad (3.3)$$

$$\alpha = 1, 2; \quad i, n = 1, 3, 5 \ldots; \quad l, m = 0, 2, 4, \ldots$$

$$E^{(1)}_{il} = A^{(2)}_{il} = E_{il}, \quad A^{(1)}_{il} = E^{(2)}_{il} = A_{il}$$

The meaning of the superscripts l, m is such that when they are zero, the given coefficients should be multiplied by $1/2$; P^{α}_{nilm}, K^{α}_{nml} and L^{α}_{nm} are the matrix elements of the system which are simple but cumbersome expressions appearing as a result of the re-expansion of some trigonometric functions in terms of a certain class of eigenfunctions, which is chosen on each face of the parallelepiped. The elements T^{α}_{nim} are somewhat more complex since the series which are the linear combinations of the matrix elements of the type mentioned above, have to be summed. Thus, the total number of unknowns which consists of 12 infinite sequences can be reduced to two infinite sequences for which the homogeneous system (3.3) was obtained.

4. Simplification of the system obtained and numerical results. Even with modern computing techniques the solution of the system (3.3) presents considerable mathematical difficulties. This can be explained both by the complexity of summing of the series making up the matrix elements T^{α}_{nim} and by the fact that the matrix of coefficients of the infinite system obtained has a three-dimensional structure. However, in a number of cases, it is possible to simplify the system. The detailed analysis of the matrix elements shows that for $y_0 \to 0$ and $z_0 \to 0$ the quantities L^{α}_{nm} and K^{α}_{nml} remain finite, whereas P^{α}_{nilm} and T^{α}_{nim} decrease as y_0^3 or z_0^3. However, if $x_0 \to \infty$, then L^{α}_{nm} and K^{α}_{nml} also remain finite, and P^{α}_{nilm} and T^{α}_{nim} decrease as $1 / x_0^3$ and as $1 / x_0^2$ in the neighborhood of the points $\varkappa'_{ml}{}' x_0 = p\pi$, where $p=0,1,2,\ldots$.

Thus, for $y_0 / x_0 \ll 1$ and $z_0 / x_0 \ll 1$ the coefficients P^{α}_{nilm} and T^{α}_{nim} can be neglected compared with L^{α}_{nm} and K^{α}_{nml}, and then the system (3.3) can be written in the following form:

$$\sum_{l}^{m} K^{\alpha}_{nml} E_{nl} + L^{\alpha}_{nm} A_{nm} = 0 \qquad (4.1)$$

$$\alpha = 1, 2; \quad n = 1, 3, 5, \ldots; \quad l, m = 0, 2, 4, \ldots$$

Letting x_0 and z_0 in (4.1) tend to zero, it can be seen that the system obtained is satisfied identically since $A_{nm} = E_{nm} = 0$ for $m \neq 0$. For $m = l = 0$, we have

$$\frac{\lambda}{\sigma_{n0} y_0} E_{n0} + \left[\left(\lambda + \frac{2\mu}{k^2} \sigma^2_{n0} \right) \operatorname{ctg} \tau_{n0} z_0 + \frac{4\mu}{k^2} \tau'_{n0} \tau_{n0} \operatorname{ctg} \tau'_{n0} z_0 \frac{\alpha^2_{n0}}{\tau'^2_{n0} - \alpha^2_{n0}} \right] A_{n0} = 0 \qquad (4.2)$$

$$\frac{\lambda}{\tau_{n0} z_0} A_{n0} + \left[\left(\lambda + \frac{2\mu}{k^2} \sigma^2_{n0} \right) \operatorname{ctg} \sigma_{n0} y_0 + \frac{4\mu}{k^2} \sigma'_{n0} \sigma_{n0} \operatorname{ctg} \sigma'_{n0} y_0 \frac{\beta^2_{n0}}{\sigma'^2_{n0} - \beta^2_{n0}} \right] E_{n0} = 0$$

By equating the determinant of this system to zero, we obtain an equation for the frequencies. This equation was solved for k^2 and the frequency was found using the formula

$$\omega = 2\pi F = vk \sqrt{\frac{1-\sigma}{(1+\sigma)(1-2\sigma)}} \tag{4.3}$$

where σ is Poisson's ratio and v is the velocity of propagation of the oscillations.

The natural frequencies for parallelepipeds of various dimensions were calculated. The results were compared with existing experimental data [6]. Table 1 shows experimental results for F_e and those for F_c obtained from the homogeneous system (4.2) and computed by formula (4.3), for $100 \times 10 \times 10$ mm parallepiped (A) and $100 \times 4.8 \times 3.8$ mm parallelepiped (B).

Table 1

	n	1	3	5	7	9	11	15
A	F_e, kHz	24.77	73.82	121.2	164.9	201.9	230.6	266.1
	F_c, kHz	24.76	73.86	121.4	165.7	203.3	223.4	273.3
						216.9	231.9	411.2
B	F_e, kHz	22.44	67.17	111.8	156.2	200.1	243.3	325.9
	F_c, kHz	22.44	67.28	112.0	156.5	200.5	244.1	327.9

As can be seen from Table 1, the error in determining the frequencies varies from 0.04% to 3% depending on the number of eigenvalues and on the dimensions of a parallelepiped. Apart from that, in the first case, beginning from $n = 9$, the calculations give two frequencies for each n. This can apparently be explained by the fact that for the given geometry, the high eigenvalues must be found by solving the system (3.3) and not from (4.2).

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